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Wavelet bases in generalized Besov spaces ☆

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Abstract

In this paper we obtain a wavelet representation in (inhomogeneous) Besov spaces of generalized smoothness via interpolation techniques. As consequence, we show that compactly supported wavelets of Daubechies type provide an unconditional Schauder basis in these spaces when the integrability parameters are finite.

Keywords: Besov spaces; Generalized smoothness; Real interpolation; Wavelet representation; Wavelet bases

1. Introduction

Wavelets have many applications into mathematics and other areas, such as engineering and physics. For instance, wavelet bases are used in the numerical resolution of some PDE's with the advantage of providing fast and efficient algorithms. Concerning functions spaces, wavelet bases give us the possibility of describing their elements in terms of basic and simple "building blocks." In general, an important point is that we can characterize the original (quasi-)norm by means of certain sums involving the wavelet coefficients. On the other hand, wavelet bases can be quite useful to study some intrinsic questions related to

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functions spaces. Recently, for example, they were successfully used to estimate entropy numbers of compact embeddings between weighted spaces (see [12] for details).

Motivated by a recent work of Triebel on wavelet bases in function spaces, we deal with wavelet representations in Besov spaces with generalized smoothness. In [16] it was proved, in particular, that compactly supported wavelets of Daubechies type form an unconditional Schauder basis in the "classic" Besov spaces B^s_{pq}. Our main aim is to extend this result to the "generalized" Besov spaces B^{ϕ}_{pq} (cf. Definition 2), showing that the same wavelet system also provides an unconditional Schauder basis in these spaces. We would like to remark that function spaces of generalized smoothness have applications in other fields such as probability theory and stochastic processes (see [7]).

We realize that it is possible to get the result without repeating the approach suggested in [16]. Hence, instead of making use of all that powerful tools (atomic decompositions, local means, maximal functions, duality theory), we try mainly to take advantage of the classic case by means of suitable interpolation techniques. We would like to remark that interpolation tools were recently used by Caetano (see [3]) in order to get subatomic representations of Bessel potential spaces modelled on Lorentz spaces from the corresponding ones for the usual spaces H^s_p.

As long as wavelet bases literature is concerned, we refer to [6,14,22] and, of course, to [16] as well as to other references therein. Close to this matter we also mention [10], where wavelet decompositions of Besov spaces were studied in a multiresolution analysis framework.

This paper is structured as follows. In Section 2 we give the definition of Besov spaces of generalized smoothness and compare them to other well-known function spaces. In this section we also discuss some interpolation properties which will play a key role later on. Section 3 is devoted to the wavelet representation of Besov spaces. For convenience, we contextualize the problem recalling what is already done in the "classic" case, and then we formulate our main result as well as some of its consequences.

We will follow standard notation or it will be properly introduced whenever needed.

2. Generalized Besov spaces

2.1. Preliminaries

We shall consider standard notation as follows. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $\mathbb{Z}^n$ the usual lattice of all points with integers components ($n \in \mathbb{N}$). For $0 < p \leqslant \infty$, $L_p(\mathbb{R}^n)$ denotes the well-known quasi-Banach space with respect to the Lebesgue measure, quasi-normed by

$$\| f \mid L_p(\mathbb{R}^n) \| = \left(\int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{1/p},$$

with the usual modification if $p = \infty$. Let $C(\mathbb{R}^n)$ be the space of all complex-valued uniformly continuous bounded functions in $\mathbb{R}^n$ and let, for $r \in \mathbb{N}$,

$$C^r(\mathbb{R}^n) = \{ f \in C(\mathbb{R}^n): \ D^\alpha f \in C(\mathbb{R}^n), \ |\alpha| \leqslant r \}, \tag{1}$$

normed by

$$\|f \mid C^r(\mathbb{R}^n)\| = \sum_{|\alpha|\leqslant r} \|D^\alpha f \mid L_\infty(\mathbb{R}^n)\|.$$

By $\mathcal{S}(\mathbb{R}^n)$ we denote the Schwartz space of all rapidly decreasing and infinitely differentiable functions on $\mathbb{R}^n$, and by $\mathcal{S}'(\mathbb{R}^n)$ its topological dual, that is, the space of all tempered distributions. If $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then $\mathcal{F}\varphi$ (or $\hat{\varphi}$) stands for the Fourier transform of φ,

$$(\mathcal{F}\varphi)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi}\varphi(x)\,dx, \quad \xi \in \mathbb{R}^n, \tag{2}$$

whereas $\mathcal{F}^{-1}\varphi$ (or $\varphi^\vee$) denotes its inverse Fourier transform, given by the right-hand side of (2) with i in place of $-i$. Both the Fourier transform and its inverse are extended to $\mathcal{S}'(\mathbb{R}^n)$ in the usual way.

Let $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ be such that

$$\varphi_0(x) = 1 \quad \text{if } |x| \leqslant 1 \quad \text{and} \quad \operatorname{supp}\varphi_0 \subset \{x \in \mathbb{R}^n\colon\ |x| \leqslant 2\}. \tag{3}$$

Putting

$$\varphi_1(x) := \varphi_0(x/2) - \varphi_0(x) \quad \text{and} \quad \varphi_j(x) := \varphi_1(2^{-j+1}x), \quad x \in \mathbb{R}^n,\ j \in \mathbb{N}, \tag{4}$$

then

$$\operatorname{supp}\varphi_j \subset \{x \in \mathbb{R}^n\colon\ 2^{j-1} \leqslant |x| \leqslant 2^{j+1}\}, \quad j \in \mathbb{N},$$

and

$$\sum_{j=0}^\infty \varphi_j(x) = 1, \quad x \in \mathbb{R}^n.$$

Hence $\{\varphi_j\}_{j\in\mathbb{N}_0}$ forms a dyadic smooth resolution of unity. We recall that, for $s \in \mathbb{R}$, $0 < p \leqslant \infty$, $0 < q \leqslant \infty$, the usual Besov and Triebel–Lizorkin spaces are defined as the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f \mid B^s_{pq}(\mathbb{R}^n)\| := \left(\sum_{j=0}^\infty 2^{jsq} \|(\varphi_j \hat{f})^\vee \mid L_p(\mathbb{R}^n)\|^q\right)^{1/q} \tag{5}$$

and

$$\|f \mid F^s_{pq}(\mathbb{R}^n)\| := \left\| \left(\sum_{j=0}^\infty 2^{jsq} |(\varphi_j \hat{f})^\vee(\cdot)|^q\right)^{1/q} \,\Big|\, L_p(\mathbb{R}^n) \right\| \tag{6}$$

(with the usual modification if $q = \infty$ and $p < \infty$ in the F-case) are finite, respectively. They are quasi-Banach spaces and are independent of the system $\{\varphi_j\}_{j\in\mathbb{N}_0}$ chosen according to (3) and (4) (with equivalent quasi-norms). We refer to [18] for a systematic theory on these spaces. It is well known that these scales contain some classic spaces as special cases. For instance,

$$F^s_{p,2}(\mathbb{R}^n) = H^s_p(\mathbb{R}^n), \quad s \in \mathbb{R},\ 1 < p < \infty,$$

are the *fractional Sobolev spaces* (they are the *classic Sobolev spaces* when $s \in \mathbb{N}_0$) and

$$F^0_{p,2}(\mathbb{R}^n) = h_p(\mathbb{R}^n), \quad 0 < p < \infty, \tag{7}$$

are the *local* (or *inhomogeneous*) *Hardy spaces* introduced by Goldberg (see [11]).

In the sequel, we need to deal with some sequence spaces into a general context as follows. Let E be a quasi-normed space, I a countable set and $0 < q \leqslant \infty$. We denote by $\ell_q(I, E)$ the "sequence" spaces of all E-valued families $a \equiv \{a_i\}_{i \in I}$ such that $\|a \mid \ell_q(I, E)\|$ is finite, where

$$\|a \mid \ell_q(I, E)\| := \left(\sum_{i \in I} \|a_i \mid E\|^q \right)^{1/q}, \quad 0 < q < \infty, \tag{8}$$

and

$$\|a \mid \ell_\infty(I, E)\| := \sup_{i \in I} \|a_i \mid E\| \tag{9}$$

define quasi-norms. If the set I is clear from the context, we shall omit it. Besides, we may omit E from the notation if $E = \mathbb{C}$.

2.2. Definition and basic properties

Roughly speaking, we obtain Besov spaces of generalized smoothness replacing the usual regularity index s in (5) by a certain function with given properties. We consider a sufficiently wide class of such functions, which allows us to cover many cases in the literature.

Definition 1. We say that a function $\phi : (0, \infty) \to (0, \infty)$ belongs to the class $\mathcal{B}$ if it is continuous, $\phi(1) = 1$, and

$$\bar{\phi}(t) := \sup_{s>0} \frac{\phi(ts)}{\phi(s)} < \infty, \quad t \in (0, \infty).$$

We refer to [5,13] for more details concerning this class. For a function $\phi \in \mathcal{B}$, the *Boyd upper* and *lower indices* $\alpha_{\bar{\phi}}$ and $\beta_{\bar{\phi}}$ are then well-defined, respectively, by

$$\alpha_{\bar{\phi}} = \lim_{t \to +\infty} \frac{\log \bar{\phi}(t)}{\log t} \quad \text{and} \quad \beta_{\bar{\phi}} = \lim_{t \to 0} \frac{\log \bar{\phi}(t)}{\log t}$$

with $-\infty < \beta_{\bar{\phi}} \leqslant \alpha_{\bar{\phi}} < +\infty$.

If E is a quasi-normed space, $0 < q \leqslant \infty$ and $\phi \in \mathcal{B}$, one can consider the spaces $\ell^\phi_q(E)$ of all sequences $\{a_j\}_{j \in \mathbb{N}_0}$ such that $\{\phi(2^j) a_j\}_{j \in \mathbb{N}_0} \in \ell_q(E)$, equipped with the quasi-norms $\|\cdot \mid \ell_q(E)\|$ according to (8) and (9) (with $I = \mathbb{N}_0$). When $\phi(t) = t^s$, $t \in (0, \infty)$, $s \in \mathbb{R}$, we simply write $\ell^s_q(E)$ instead of $\ell^\phi_q(E)$ for short.

Let $\{\varphi_j\}_{j \in \mathbb{N}_0} \subset \mathcal{S}(\mathbb{R}^n)$ be a system with the following properties:

$$\operatorname{supp} \varphi_0 \subset \{\xi \in \mathbb{R}^n \colon |\xi| \leqslant 2\}; \tag{10}$$

$$\operatorname{supp} \varphi_j \subset \{\xi \in \mathbb{R}^n \colon 2^{j-1} \leqslant |\xi| \leqslant 2^{j+1}\}, \quad j \in \mathbb{N}; \tag{11}$$

$$\sup_{\xi\in\mathbb{R}^n}\left|D^\alpha\varphi_j(\xi)\right| \leqslant c_\alpha 2^{-j|\alpha|}, \quad j\in\mathbb{N}_0,\ \alpha\in\mathbb{N}_0^n; \tag{12}$$

$$\sum_{j=0}^{\infty}\varphi_j(\xi)=1, \quad \xi\in\mathbb{R}^n. \tag{13}$$

Definition 2. Let $\{\varphi_j\}_{j\in\mathbb{N}_0}$ be a dyadic resolution of unity with the properties (10)–(13) above. For $\phi\in\mathcal{B}$, $0<p\leqslant\infty$, and $0<q\leqslant\infty$, we define $B^\phi_{pq}(\mathbb{R}^n)$ as being the class of all $f\in\mathcal{S}'(\mathbb{R}^n)$ such that $\{(\varphi_j\hat{f})^\vee\}_{j\in\mathbb{N}_0}\in\ell^\phi_q(L_p(\mathbb{R}^n))$ with

$$\left\| f \mid B^\phi_{pq}\left(\mathbb{R}^n\right)\right\| := \left\|\left\{(\varphi_j\hat{f})^\vee\right\}_{j\in\mathbb{N}_0} \mid \ell^\phi_q\left(L_p\left(\mathbb{R}^n\right)\right)\right\|.$$

These spaces were studied by Merucci (see [13]) as a result of real interpolation with function parameter between Sobolev spaces and then by Cobos and Fernandez in [5]. Such as in the classic case according to (5), they are quasi-Banach spaces and are independent of the system $\{\varphi_j\}_{j\in\mathbb{N}_0}$ chosen, up to equivalent quasi-norms. We point out that the spaces $B^s_{pq}(\mathbb{R}^n)$ can be obtained as a particular case of the spaces $B^\phi_{pq}(\mathbb{R}^n)$ by taking $\phi(t)=t^s$, $t\in(0,\infty)$, $s\in\mathbb{R}$.

In general, we are only dealing with functions spaces on $\mathbb{R}^n$. Hence, from now on, we shall omit the $\mathbb{R}^n$ in their notation. For convenience, we will refer to the spaces B^s_{pq} as *classic* Besov spaces.

Besov spaces with generalized smoothness have been considered and studied by many authors in different contexts. We refer to the paper [7] for historical remarks and literature concerning this subject. In [7] we can also find a general and unified approach for these spaces, as well as the counterpart for the Triebel–Lizorkin scale. As far as Besov spaces are concerned, it is possible to define generalized spaces B^σ_{pq} by replacing $\phi(2^j)$ by σ_j, $j\in\mathbb{N}_0$, in Definition 2, where σ is a certain *admissible sequence* of positive real numbers in the sense of [7]:

$$B^\sigma_{pq}=\left\{f\in\mathcal{S}':\ \left\|f \mid B^\sigma_{pq}\right\| := \left\|\left\{\sigma_j(\varphi_j\hat{f})^\vee\right\}_{j\in\mathbb{N}_0} \mid \ell_q(L_p)\right\|<\infty\right\}, \tag{14}$$

where $\sigma\equiv\{\sigma_j\}_{j\in\mathbb{N}_0}$ satisfies the condition

$$d_0\sigma_j\leqslant\sigma_{j+1}\leqslant d_1\sigma_j, \quad \forall j\in\mathbb{N}_0, \tag{15}$$

for some $d_0, d_1>0$. The definition given in [7] is even more general: it is introduced a fourth parameter $N\equiv\{N_j\}_{j\in\mathbb{N}_0}$ related to generalized resolutions of unity, namely, allowing different sizes for the support of the involved functions. We restrict ourselves here to the standard decomposition, that is, with $N=\{2^j\}_{j\in\mathbb{N}_0}$.

Some "other" generalized spaces of Besov type were introduced by Edmunds and Triebel. They are usually denoted by $B^{(s,\Psi)}_{pq}$ and are defined as in (5) with $2^{jsq}\,\Psi(2^{-j})^q$ in place of 2^{jsq}. The parameter Ψ here represents a perturbation on the smoothness index s, and, of course, it fulfills certain conditions. We refer to [15] for a systematic study on spaces $B^{(s,\Psi)}_{pq}$.

As it was remarked in [7], the spaces $B^{(s,\Psi)}_{pq}$ are covered by the general formulation (14), by taking $\sigma_j=2^{js}\Psi(2^{-j})$, $j\in\mathbb{N}_0$. Since we have $\bar{\phi}(1/2)^{-1}\phi(2^j)\leqslant\phi(2^{j+1})\leqslant\bar{\phi}(2)\phi(2^j)$, $j\in\mathbb{N}_0$, the spaces B^ϕ_{pq}, $\phi\in\mathcal{B}$, are also a particular case of the spaces defined in (14). However, we would like to point out that is enough to consider the spaces B^{ϕ}_{pq}. This fact may be justified by the following result, which was suggested to us by Caetano.

Proposition 3. *Let σ be an admissible sequence in the sense of* (15) *and* $0 < p, q \leqslant \infty$. *Then there exists a function $\phi_\sigma \in \mathcal{B}$ such that*

$$B^{\phi_\sigma}_{pq} = B^{\sigma}_{pq}.$$

Proof. Let σ be admissible. First, we remark that one can always assume $\sigma_0 = 1$ without loss the generality. In fact, the sequence σ' defined as $\sigma'_0 = 1$ and $\sigma'_j = \sigma_j$, $j \in \mathbb{N}$, is equivalent to σ, so $B^{\sigma}_{pq} = B^{\sigma'}_{pq}$.
We can construct a function $\phi_\sigma \in \mathcal{B}$ as follows:

$$\phi_\sigma(t) = \begin{cases} \frac{\sigma_{j+1}-\sigma_j}{2^j}(t - 2^j) + \sigma_j, & t \in [2^j, 2^{j+1}),\ j \in \mathbb{N}_0, \\ \sigma_0, & t \in (0, 1) \end{cases}$$

(cf. [4, Section 2.2]). Hence, $\phi_\sigma(2^j) = \sigma_j$ for all $j \in \mathbb{N}_0$ and we get the result. $\square$

Taking into account this proposition, from now on we will only deal with Besov spaces from Definition 2. Such as in the classic case (cf. [18, pp. 47, 48]) one proves the following embeddings related to the spaces B^{ϕ}_{pq}.

Proposition 4.

(i) *Let $\phi \in \mathcal{B}$, $0 < p \leqslant \infty$, $0 < q \leqslant \infty$. Then*

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^{\phi}_{pq}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n).$$

(ii) *Let $\phi \in \mathcal{B}$, $0 < p \leqslant \infty$, $0 < q_0 \leqslant q_1 \leqslant \infty$. Then*

$$B^{\phi}_{pq_0}(\mathbb{R}^n) \hookrightarrow B^{\phi}_{pq_1}(\mathbb{R}^n).$$

(iii) *Let $\phi, \psi \in \mathcal{B}$, $0 < p \leqslant \infty$, $0 < q_0, q_1 \leqslant \infty$. If $\left\{\frac{\phi(2^j)}{\psi(2^j)}\right\}_{j\in\mathbb{N}_0} \in \ell_{\min\{q_1,1\}}$, then*

$$B^{\psi}_{pq_0}(\mathbb{R}^n) \hookrightarrow B^{\phi}_{pq_1}(\mathbb{R}^n).$$

As usually, the symbol "$\hookrightarrow$" above indicates that the corresponding embedding is continuous. Property (iii) is important, in particular, to derive Lemma 6 bellow.

2.3. Interpolation with function parameter

As it was referred before, the spaces B^{ϕ}_{pq}, $\phi \in \mathcal{B}$, can be obtained from real interpolation between Sobolev spaces with an appropriate function parameter. Interpolation of this kind fits well into these generalized Besov spaces framework if the function parameter belongs to the same class $\mathcal{B}$. We refer to the papers [5,13] for the notation and basic properties concerning interpolation. In [5] several interpolation results were obtained for the spaces B^{ϕ}_{pq} in the Banach case ($1 \leqslant p, q \leqslant \infty$). The approach followed there was based on interpolation properties of sequence spaces. Those properties were then transferred to the spaces B^{ϕ}_{pq}, by means of the so-called *method of retraction and co-retraction* (cf. [1, p. 150] and [19, p. 22], for example). Briefly, let E be a quasi-Banach space, $\phi \in \mathcal{B}$ and $0 < q_0, q_1, q \leqslant \infty$. Taking into account [5, Theorem 5.1 and Remark 5.4], one can write

$$\left(\ell^{s_0}_{q_0}(E), \ell^{s_1}_{q_1}(E)\right)_{\gamma,q} = \ell^{\phi}_q(E) \tag{16}$$

if $s_0, s_1 \in \mathbb{R}$ with $s_1 < \beta_{\bar{\phi}} \leqslant \alpha_{\bar{\phi}} < s_0$ and

$$\gamma(t) = \frac{t^{\frac{s_0}{s_0-s_1}}}{\phi\left(t^{\frac{1}{s_0-s_1}}\right)}, \quad t \in (0,\infty). \tag{17}$$

It is possible to show that B^{ϕ}_{pq} is a retract of $\ell^{\phi}_q(L_p)$ if $p \geqslant 1$ by constructing certain applications (retraction and co-retractions) based on the Fourier transform. But this does not work if $0 < p < 1$. However, as it was remarked in [5, Remark 5.4], some of the interpolation results obtained hold in the quasi-Banach case as well. We do not intend to go into too many details, but we give here a brief description how this question in the general case can be dealt with. Following [17, Theorem 2.2.10], one can prove the result bellow.

Proposition 5. *Let $f \in \mathcal{S}'$, $0 < p < \infty$ and $\{\varphi_j\}_{j\in\mathbb{N}_0}$ satisfying the conditions* (10)–(13). *Then $\mathcal{F}^{-1}(\varphi_j \mathcal{F} f) \in L_p$ if and only if $\mathcal{F}^{-1}(\varphi_j \mathcal{F} f) \in h_p$, $j \in \mathbb{N}_0$. Moreover, there are constants $c_1, c_2 > 0$ independent of f and j such that*

$$c_1 \left\| \mathcal{F}^{-1}(\varphi_j \mathcal{F} f) \,\middle|\, L_p \right\| \leqslant \left\| \mathcal{F}^{-1}(\varphi_j \mathcal{F} f) \,\middle|\, h_p \right\| \leqslant c_2 \left\| \mathcal{F}^{-1}(\varphi_j \mathcal{F} f) \,\middle|\, L_p \right\|.$$

Note that h_p is the local Hardy space from (7). Using these estimates, one can replace L_p by h_p in Definition 2 when $p < \infty$ (note that $h_p = L_p$ if $1 < p < \infty$). With this change, one avoids the mentioned troubles caused by the Fourier transform. On the other hand, we can prove that B^{ϕ}_{pq} is a retract of $\ell^{\phi}_q(h_p)$: if $\{\varphi_j\}_{j\in\mathbb{N}_0} \subset \mathcal{S}$ is a system with the properties (10)–(13), then

$$R\{f_j\}_{j\in\mathbb{N}_0} := \sum_{j=0}^{\infty} \mathcal{F}^{-1}(\tilde{\varphi}_j \, \mathcal{F} f_j), \quad \text{with } \tilde{\varphi}_j = \sum_{r=-1}^{1} \varphi_{j+r},$$

is a retraction from $\ell^{\phi}_q(h_p)$ to B^{ϕ}_{pq} and $Sf := \{\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)\}_{j\in\mathbb{N}_0}$ is the corresponding co-retraction. We remark that R is well-defined with the help of the following lemma, which can be proved following similar techniques as in [23, Theorem 3.6],

Lemma 6. *Let $\phi \in \mathcal{B}$, $0 < p \leqslant 1$, $0 < q \leqslant \infty$. Assume that $\{g_j\}_{j\in\mathbb{N}_0} \subset \mathcal{S}'$ fulfills the conditions*

$$\operatorname{supp} \mathcal{F} g_0 \subset \left\{x\colon |x| \leqslant 2\right\} \quad \textit{and} \quad \operatorname{supp} \mathcal{F} g_j \subset \left\{x\colon 2^{j-1} \leqslant |x| \leqslant 2^{j+1}\right\}, \quad j \in \mathbb{N}.$$

If $\|\phi(2^j) g_j \mid \ell_q(h_p)\| < \infty$, then $\sum_{j=0}^{\infty} g_j$ converges in $\mathcal{S}'$.

Hence, taking into account the remarks above, it is possible to get the result bellow, which will play a crucial role in next section.

Proposition 7. *Let $\phi \in \mathcal{B}$, $0 < p \leqslant \infty$, and $0 < q_0, q_1, q \leqslant \infty$. Assume $s_0, s_1 \in \mathbb{R}$ satisfy $s_1 < \beta_{\bar{\phi}} \leqslant \alpha_{\bar{\phi}} < s_0$ and γ as in (17). Then*

$$\left(B^{s_0}_{pq_0}, B^{s_1}_{pq_1}\right)_{\gamma,q} = B^{\phi}_{pq}.$$

Proposition 7 shows, in particular, that spaces $B^{(s,\Psi)}_{pq}$ mentioned in Section 2.2 can be obtained by interpolation of classic Besov spaces with a suitable function parameter. This fact was already observed in [2].

3. Wavelet representation of Besov spaces

The aim of this section is to obtain wavelet representations for the generalized Besov spaces under consideration. We will make use of the system considered in [16] and follow the same notation.

Let $L_j = L = 2^n - 1$ if $j \in \mathbb{N}$ and $L_0 = 1$. It is known that, for any $r \in \mathbb{N}$, there are real compactly supported functions

$$\psi_0 \in C^r, \qquad \psi^l \in C^r, \quad l = 1, \ldots, L, \tag{18}$$

with

$$\int_{\mathbb{R}^n} x^\alpha \psi^l(x)\,dx = 0, \quad \alpha \in \mathbb{N}_0^n,\ |\alpha| \leqslant r, \tag{19}$$

such that

$$\left\{2^{jn/2}\psi^l_{jm}\colon\ j \in \mathbb{N}_0,\ 1 \leqslant l \leqslant L_j,\ m \in \mathbb{Z}^n\right\} \tag{20}$$

with

$$\psi^l_{jm}(\cdot) = \begin{cases} \psi_0(\cdot - m), & j = 0,\ m \in \mathbb{Z}^n,\ l = 1, \\ \psi^l(2^{j-1}\cdot - m), & j \in \mathbb{N},\ m \in \mathbb{Z}^n,\ 1 \leqslant l \leqslant L, \end{cases} \tag{21}$$

is an orthonormal basis in L_2. As mentioned in [16], an example of such a system of functions is the (inhomogeneous) Daubechies wavelet basis (see, for example, [6,14,22] for further information).

Wavelets with the properties above are sufficiently good to provide unconditional bases in many classical spaces. For instance, it was known that the mentioned Daubechies system forms an unconditional Schauder basis in the Sobolev spaces H^s_p if $1 < p < \infty$, $r > |s|$, and in the Besov spaces B^s_{pq} if $1 \leqslant p, q < \infty$, $r > |s|$. These two examples show, in particular, that the smoothness required on the wavelets in (18) should be large enough, depending on the regularity of the functions that we pretend to represent. This fact can also be observed in the sequel.

3.1. The classic case

The main aim in [16] was to extend the results above about Sobolev spaces and some Besov spaces to the entire scales B^s_{pq} and F^s_{pq}. For convenience, we recall here the main result related to Besov spaces. Let $I = \{(l, j, m)\colon\ j \in \mathbb{N}_0,\ 1 \leqslant l \leqslant L_j,\ m \in \mathbb{Z}^n\}$ and $I' = \{(l, j)\colon\ j \in \mathbb{N}_0,\ 1 \leqslant l \leqslant L_j\}$.

Theorem 8. *Let $s \in \mathbb{R}$, $0 < p \leqslant \infty$, $0 < q \leqslant \infty$, and*

$$r(s, p) := \max\left(s, \frac{2n}{p} + \frac{n}{2} - s\right). \tag{22}$$

(i) *Assume $r \in \mathbb{N}$ with $r > r(s, p)$ and let $f \in \mathcal{S}'$. Then $f \in B^s_{pq}$ if and only if it can be represented as*

$$f = \sum_{(l,j,m)\in I} \lambda^l_{jm} \psi^l_{jm} \quad \text{with } \lambda \in b^s_{pq}, \tag{23}$$

unconditional convergence in $\mathcal{S}'$ and in any space B^t_{pu} if $t < s$. Moreover, the representation (23) *is unique*:

$$\lambda = \lambda(f) \quad \text{with } \lambda^l_{jm}(f) := 2^{jn}\langle f, \psi^l_{jm}\rangle. \tag{24}$$

Furthermore, $f \mapsto \{2^{jn}\langle f, \psi^l_{jm}\rangle\}_{(l,j,m)\in I}$ defines an isomorphic map of B^s_{pq} onto b^s_{pq} and

$$\left\| f \mid B^s_{pq} \right\| \sim \left\| \lambda(f) \mid b^s_{pq} \right\| \tag{25}$$

(*equivalent quasi-norms*).

(ii) *In addition, if* $\max(p, q) < \infty$, *then* (23) *with* (24) *converges unconditionally in B^s_{pq} and $\{\psi^l_{jm}\}_{(l,j,m)\in I}$ is an unconditional Schauder basis in B^s_{pq}.*

Here, b^s_{pq} is the space of all complex-valued sequences $\lambda \equiv \{\lambda^l_{jm}\}_{(l,j,m)\in I}$ such that

$$\left\| \lambda \mid b^s_{pq} \right\| := \left(\sum_{(l,j)\in I'} 2^{j(s-n/p)q} \left(\sum_{m\in\mathbb{Z}^n} \left|\lambda^l_{jm}\right|^p \right)^{q/p} \right)^{1/q} < \infty,$$

with standard modifications if $p = \infty$ and/or $q = \infty$.

Remark 9. This result was extended to weighted function spaces of Besov and Triebel–Lizorkin type very recently in the paper [12].

Remark 10. The symbol $\langle f, \psi^l_{jm}\rangle$ in (24), $f \in B^s_{pq}$, $\psi^l_{jm} \in C^r$, should be properly interpreted since the functions ψ^l_{jm} are not in $\mathcal{S}$ in general. As remarked in [16], it makes sense if $r > -s + \sigma_p$ (with $\sigma_p := \max(n/p - n, 0)$), which is covered by condition $r > r(s, p)$. In fact, in that case, one can interpret f as an element of the dual of a space to which ψ^l_{jm} belongs to.

3.2. The generalized case

The proof of Theorem 8 was based on atomic decompositions, characterizations by local means, and duality theory (we refer to [20,21] for details on these properties). An important point there was that the wavelets considered were simultaneously atoms and kernels of those local means. It was also commented in [16] the possibility of getting a similar result in the context of other scales of function spaces. To do that, it would be enough to have the same tools available. However, as we mentioned before, we will not follow this approach. Instead, we will consider a scheme based on interpolation techniques in order to take advantage of the already known wavelet decompositions for the classic case.

Let $\phi \in \mathcal{B}$, $0 < p \leqslant \infty$, $0 < q \leqslant \infty$. For our purposes we need to introduce the sequence spaces b^{ϕ}_{pq}, consisting of all complex-valued sequences $\lambda \equiv \{\lambda^l_{jm}\}_{(l,j,m)\in I}$ such that the quasi-norm

$$\|\lambda \mid b^{\phi}_{pq}\| := \left(\sum_{(l,j)\in I'} \left(\phi\left(2^j\right) 2^{-jn/p}\right)^q \left(\sum_{m\in\mathbb{Z}^n} \left|\lambda^l_{jm}\right|^p \right)^{q/p} \right)^{1/q} \tag{26}$$

(with the usual modifications if $p = \infty$ and/or $q = \infty$) is finite. When $\phi(t) = t^s$, $t \in (0,\infty)$, $s \in \mathbb{R}$, then b^{ϕ}_{pq} coincides with the space b^s_{pq} defined in [16]. We would like to remark that sequence spaces with this structure were introduced by Frazier and Jawerth in [8,9] in connection with atomic decompositions of (classic) Besov and Triebel–Lizorkin spaces and they have been used afterwards by many authors.

The interpolation property bellow will be very useful in proving our main result.

Proposition 11. *Let $\phi \in \mathcal{B}$ and $0 < p, q, q_0, q_1 \leqslant \infty$. If s_0, s_1 are real numbers fulfilling $s_1 < \beta_{\bar{\phi}} \leqslant \alpha_{\bar{\phi}} < s_0$, then we have*

$$\left(b^{s_0}_{pq_0}, b^{s_1}_{pq_1}\right)_{\gamma,q} = b^{\phi}_{pq},$$

where γ is defined as in (17).

Proof. Firstly, we note that spaces $b^{s_0}_{pq_0}$ and $b^{s_1}_{pq_1}$ form an interpolation couple since they are both continuously embedded in $b^{s_1}_{p\infty}$, for example. We can interpret b^{ϕ}_{pq} as the sequence space $\ell^{\phi_1}_q(\ell_p(\mathbb{Z}^n))$ where $\phi_1(t) := \phi(t)\, t^{-n/p}$, $t \in (0,\infty)$. In fact, the index l does not bring any trouble. It is not hard to see that formula (16) remains valid for these spaces. On the other hand, the Boyd indices of ϕ_1 are given by

$$\beta_{\overline{\phi_1}} = \beta_{\bar{\phi}} - \frac{n}{p} \quad \text{and} \quad \alpha_{\overline{\phi_1}} = \alpha_{\bar{\phi}} - \frac{n}{p}.$$

Taking $\sigma_i = s_i - n/p$ $(i = 0, 1)$, then we have

$$\sigma_1 < \beta_{\overline{\phi_1}} \leqslant \alpha_{\overline{\phi_1}} < \sigma_0 \quad \text{and} \quad \gamma_1(t) := t^{\frac{\sigma_0}{\sigma_0-\sigma_1}} / \phi_1\left(t^{\frac{1}{\sigma_0-\sigma_1}}\right) = \gamma(t), \quad t \in (0,\infty).$$

Hence, attending to formula (16), we have

$$\left(\ell^{\sigma_0}_{q_0}\left(\ell_p\left(\mathbb{Z}^n\right)\right), \ell^{\sigma_1}_{q_1}\left(\ell_p\left(\mathbb{Z}^n\right)\right)\right)_{\gamma,q} = \ell^{\phi_1}_q\left(\ell_p\left(\mathbb{Z}^n\right)\right),$$

that is, $(b^{s_0}_{pq_0}, b^{s_1}_{pq_1})_{\gamma,q} = b^{\phi}_{pq}$. $\square$

Lemma 12. *Let $s \in \mathbb{R}$, $0 < p < \infty$, and $0 < q \leqslant \infty$. If $\{\lambda^l_{jm}\}_{(l,j,m)\in I} \in b^s_{pq}$ and r is a natural number such that $r > \max(s, \sigma_p - s)$, then $\sum_{(l,j,m)\in I} \lambda^l_{jm}\,\psi^l_{jm}$ converges unconditionally in B^s_{pq} if $q < \infty$ and in any B^t_{pq} with $t < s$, if $q = \infty$.*

Proof. First, we assume that $q < \infty$. From properties (18), (19), and (21), we see that, for each l, the functions $2^{-j(s-n/p)}\psi^l_{jm}$ are 1_r-atoms ($j = 0$) or $(s, p)_{r,r}$-atoms ($j \in \mathbb{N}$) according to [20, Definition 13.3], ignoring constants which are independent of ℓ, j, and m. Using the Atomic Decomposition Theorem (cf. [20, pp. 75–76]), we arrive at the conclusion that there exists $c > 0$ such that the estimate

$$\left\| \sum_{(l,j,m)\in K} \lambda^l_{jm}\psi^l_{jm} \;\Big|\; B^s_{pq} \right\|^q \leqslant c \sum_{l,j} 2^{j(s-n/p)q} \left(\sum_m \left|\lambda^l_{jm}\right|^p \right)^{q/p} \tag{27}$$

holds for all finite subsets K of I (the sums on the right-hand side run over all indices (l, j) and m such that $(l, j, m) \in K$). From this estimate and from the summability of the two families of positive real numbers involved in (26), we conclude that the partial sums on the left-hand side of (27) constitute a generalized Cauchy sequence in the complete space B^s_{pq}, thus converge in this space.

Now, let $t < s$ and $q = \infty$. We reduce this case to the previous one by using the atomic decomposition result as before (with t in place of s) and remarking that $b^s_{p\infty} \hookrightarrow b^t_{pu}$ with $0 < u < \infty$. $\square$

In the sequel, we formulate our main result related to wavelet representation and some of its consequences.

Theorem 13. *Let $\phi \in \mathcal{B}$, $0 < p < \infty$, and $0 < q \leqslant \infty$. Consider the system $\{\psi^l_{jm}\}_{(l,j,m)\in I}$ as previously. Then there exists $r(\phi, p)$ such that, for any $r \in \mathbb{N}$ with $r > r(\phi, p)$, the following holds*:

Given $f \in \mathcal{S}'$, then $f \in B^\phi_{pq}$ if and only if it can be represented as

$$f = \sum_{(l,j,m)\in I} \lambda^l_{jm}\,\psi^l_{jm} \quad \text{with } \lambda \in b^\phi_{pq} \tag{28}$$

(*unconditional convergence in $\mathcal{S}'$*). *Moreover, the "wavelet coefficients" λ^l_{jm} are uniquely determined by*

$$\lambda^l_{jm} = \lambda^l_{jm}(f) := 2^{jn}\langle f, \psi^l_{jm}\rangle, \quad (l, j, m) \in I. \tag{29}$$

Further,

$$\left\| f \;\big|\; B^\phi_{pq} \right\| \sim \left\| \lambda(f) \;\big|\; b^\phi_{pq} \right\| \tag{30}$$

(*equivalent quasi-norms*), *where $\lambda(f) \equiv \{\lambda^l_{jm}(f)\}_{(l,j,m)\in I}$.*

Proof.

Step 1. Assume that $f \in \mathcal{S}'$ can be represented as

$$f = \sum_{(l,j,m)\in I} \lambda^l_{jm}\,\psi^l_{jm} \quad \text{(unconditional convergence in } \mathcal{S}'\text{)}$$

for some $\lambda \in b^{\phi}_{pq}$. Let $s_0, s_1 \in \mathbb{R}$. Attending to Lemma 12, we conclude that the operator

$$T : b^{s_0}_{p,1} + b^{s_1}_{p,1} \longrightarrow B^{s_0}_{p,1} + B^{s_1}_{p,1},$$

given by

$$T\mu = \sum_{(l,j,m)\in I} \mu^l_{jm}\,\psi^l_{jm} \quad \text{(unconditional convergence in } \mathcal{S}'\text{)},$$

is well-defined and linear if $r > \max(r(s_0, p), r(s_1, p))$, for example, where $r(s_i, p)$ ($i = 0, 1$) is given by (22). Moreover, by Theorem 8 one concludes that the restriction of T to each $b^{s_i}_{p,1}$ is a bounded linear operator into $B^{s_i}_{p,1}$. Choosing s_0, s_1 above such that $s_1 < \beta_{\bar\phi} \leqslant \alpha_{\bar\phi} < s_0$ and attending to the interpolation property and to Propositions 7 and 11, we arrive at the conclusion that the restriction of T to b^{ϕ}_{pq} is also a bounded linear operator into B^{ϕ}_{pq}. Thus, $f \in B^{\phi}_{pq}$ and

$$\|f \mid B^{\phi}_{pq}\| = \|T\lambda \mid B^{\phi}_{pq}\| \leqslant c\|\lambda \mid b^{\phi}_{pq}\|$$

for some $c > 0$ independent of λ and f.

Step 2. Now, let $f \in B^{\phi}_{pq}$. Assume that s_0, s_1, and r fulfill the same conditions as in Step 1. Consider the operator

$$S : B^{s_0}_{p,1} + B^{s_1}_{p,1} \longrightarrow b^{s_0}_{p,1} + b^{s_1}_{p,1}$$

defined by

$$Sg = \lambda(g) := \{2^{jn}(\langle g_0, \psi^l_{jm}\rangle + \langle g_1, \psi^l_{jm}\rangle)\}_{(l,j,m)\in I}, \tag{31}$$

where $g = g_0 + g_1$ with $g_i \in B^{s_i}_{p,1}$, $i = 0, 1$. Theorem 8 (and Remark 10) shows that S is well-defined, it is linear and its restriction to each $B^{s_i}_{p,1}$ is a bounded linear operator into $b^{s_i}_{p,1}$. Taking into account the interpolation property as previously, one concludes that the restriction of S to B^{ϕ}_{pq} is a bounded linear operator into b^{ϕ}_{pq} as well. Therefore,

$$\|Sf \mid b^{\phi}_{pq}\| = \|\lambda(f) \mid b^{\phi}_{pq}\| \leqslant c\|f \mid B^{\phi}_{pq}\|, \tag{32}$$

where $c > 0$ does not dependent on f. So, $\lambda(f) \in b^{\phi}_{pq}$ and hence

$$g := \sum_{(l,j,m)\in I} \lambda^l_{jm}(f)\psi^l_{jm} \tag{33}$$

(unconditional convergence in $\mathcal{S}'$) belongs to the space B^{ϕ}_{pq} by Step 1. But Theorem 8 once again allows us to conclude that TS is the identity operator, so $g = f$. But we have (by Step 1)

$$\|f \mid B^{\phi}_{pq}\| \leqslant c\|\lambda(f) \mid b^{\phi}_{pq}\|, \tag{34}$$

$c > 0$ independent of f. Therefore, equivalence (30) follows from estimates (32) and (34). It remains to show that representation (28) is unique. We do this next. Suppose that $f \in B^{\phi}_{pq}$ admits the representation

$$f = \sum_{(l,j,m)\in I} \lambda^l_{jm}\psi^l_{jm} \quad \text{with } \lambda \in b^{\phi}_{pq} \text{ (unconditional convergence in } \mathcal{S}'\text{)}.$$

Since $B^{\phi}_{pq} \hookrightarrow B^{s_0}_{p,1} + B^{s_1}_{p,1} \hookrightarrow B^{s_1}_{p,1}$ and $b^{\phi}_{pq} \hookrightarrow b^{s_0}_{p,1} + b^{s_1}_{p,1} \hookrightarrow b^{s_1}_{p,1}$ (note that $s_0 > s_1$), then $f \in B^{s_1}_{p,1}$ has the representation

$$f = \sum_{(l,j,m)\in I} \lambda^l_{jm} \psi^l_{jm} \quad \text{with } \lambda \in b^{s_1}_{pq} \text{ (unconditional convergence in } \mathcal{S}'),$$

which is unique by Theorem 8. The proof of the theorem is completed. □

Remark 14. We can choose s_0 and s_1 close enough to $\alpha_{\bar\phi}$ and $\beta_{\bar\phi}$, respectively, and take

$$r(\phi, p) := \max\left(\alpha_{\bar\phi}, \frac{2n}{p} + \frac{n}{2} - \beta_{\bar\phi}\right)$$

in Theorem 13. In fact, in that case, is possible to choose $\varepsilon > 0$ such that r is greater than $\beta_{\bar\phi}$, $\frac{2n}{p} + \frac{n}{2} - \beta_{\bar\phi} + \varepsilon$, $\alpha_{\bar\phi} + \varepsilon$, and $\frac{2n}{p} + \frac{n}{2} - \alpha_{\bar\phi}$. On the other hand, one can take s_0 and s_1 such that

$$\beta_{\bar\phi} - \varepsilon < s_1 < \beta_{\bar\phi} \leqslant \alpha_{\bar\phi} < s_0 < \alpha_{\bar\phi} + \varepsilon.$$

In this way, we have $r > \max(r(s_0, p), r(s_1, p))$ as required in the proof above.

Remark 15. We would like to remark also that one did not make a direct use of duality results about the spaces B^{ϕ}_{pq}. According to (31), we have defined the symbol $\langle f, \psi^l_{jm}\rangle$ in (29) as the sum of two quantities interpreted as in Remark 10. Of course, taking into account the choice of s_0 and s_1 made above (which implies $B^{\phi}_{pq} \hookrightarrow B^{s_1}_{p,1}$), one can take $f_0 = 0$ and $f_1 = f$, so $\langle f, \psi^l_{jm}\rangle$ can be evaluated as described in that remark.

Corollary 16. *Let ϕ, p, and q be as in Theorem* 13. *If $r \in \mathbb{N}$ is large enough and $q < \infty$, then $\{\psi^l_{jm}\}_{(l,j,m)\in I}$ forms an unconditional Schauder basis in B^{ϕ}_{pq}.*

Proof. Attending to Theorem 13, all we need to do is to check that the series in (28) converges unconditionally in B^{ϕ}_{pq} (if $p, q < \infty$). We proceed as in the first part of the proof of Lemma 12: observe that $\phi(2^j)^{-1} 2^{jn/p} \psi^l_{jm}$ are 1_r-N-atoms ($l = 1$, $j = 0$) or $(\sigma, p)_{r,r}$-N-atoms ($j \in \mathbb{N}$) according to [7, Definition 4.4.1], with $\sigma = \{\phi(2^j)\}_{j\in\mathbb{N}_0}$ and $N = \{2^j\}_{j\in\mathbb{N}_0}$. Hence, it is possible to use the Atomic Decomposition Theorem from [7, Section 4.4.2], in order to get the counterpart of estimate (27), that is,

$$\left\| \sum_{(l,j,m)\in K} \lambda^l_{jm} \psi^l_{jm} \,\Big|\, B^{\phi}_{pq} \right\|^q \leqslant c \sum_{l,j} \left(\phi\left(2^j\right) 2^{-jn/p}\right)^q \left(\sum_m \left|\lambda^l_{jm}\right|^p\right)^{q/p}$$

with $c > 0$ independent of K. To do that we have to assume that $r > r(\phi, p)$ satisfies also the conditions mentioned in that theorem restricted to our particular case. We conclude now as in Lemma 12. □

Corollary 17. *Let ϕ, p, q, and r as in Theorem* 13. *Then*

$$\mathcal{I} : f \longmapsto \left\{2^{jn}\left\langle f, \psi^l_{jm}\right\rangle\right\}_{(l,j,m)\in I}$$

establishes a topological isomorphism from B^{ϕ}_{pq} onto b^{ϕ}_{pq} (interpreted as in Remark 15).

Proof. This result follows at once from the properties of the operators T and S studied in the proof of Theorem 13. $\square$

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